Assignment 7

Hand in no. 2, 3, 5, 9, and 10 by October 31, 2023.

1. Consider maps from $\mathbb{R}$ to itself. Provide explicit examples of continuous maps with exactly one, two and three fixed points.

2. Show that the equation $x = \dfrac{1}{2}\cos^2 x$ has a unique solution in $\mathbb{R}$.

3. Let T be a continuous map on the complete metric space X. Suppose that for some k, T^k becomes a contraction. Show that T admits a unique fixed point. This generalizes the contraction mapping principle in the case $k = 1$.

4. Show that the equation $2x\sin x - x^4 + x = 0.001$ has a root near $x = 0$.

5. Can you solve the system of equations
$$x + y^4 = 0, \quad y - x^2 = 0.015 \ ?$$

6. Can you solve the system of equations
$$x + y - x^2 = 0, \quad x - y + xy\sin y = -0.002 \ ?$$
Hint: Put the system in the form $x + \cdots = 0, \quad y + \cdots = 0$, first.

7. Let $A = \{a_{ij}\}$ be an $n \times n$ matrix. Show that
$$|Ax| \le \sqrt{\sum_{i,j} a_{ij}^2}\ |x|.$$

8. Let $A = (a_{ij})$ be an $n \times n$ matrix. Show that the matrix $I + A$ is invertible if $\sum_{i,j} a_{ij}^2 < 1$. Give an example showing that $I + A$ could become singular when $\sum_{i,j} a_{ij}^2 = 1$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and $f(x_0) = 0, f'(x_0) \ne 0$. Show that there exists some $\rho > 0$ such that
$$Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0 - \rho, x_0 + \rho),$$
is a contraction. This provides a justification for Newton's method in finding roots for an equation.

10. Consider the iteration
$$x_{n+1} = \alpha x_n(1 - x_n), \ x_0 \in [0, 1] \ .$$
Find
   (a) The range of α so that $\{x_n\}$ remains in $[0, 1]$.
   (b) The range of α so that the iteration has a unique fixed point 0 in $[0, 1]$.
   (c) Show that for $\alpha \in [0, 1]$ the fixed point 0 is attracting in the sense: $x_n \to 0$ whenever $x_0 \in [0, 1]$.